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Difficulties with the inverse scattering transform method in quantum field theory

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Abstract. We discuss some difficulties which arise in recent proposals to extend the inverse scattering transform method to nonlinear quantum field theory. In particular, for the nonlinear Schrödinger equation, we show that the usual classical methods for obtaining the Poisson brackets of the scattering data reach an impasse if they are extended to find the commutators of the corresponding quantum operators.

1. Introduction

Exactly soluble nonlinear models are rare in mathematical physics; consequently their study is a valuable tool in developing general methods and testing the results of various approximation schemes. Since the advent of the 'inverse scattering transform' (IST) method for the solution of a class of nonlinear partial differential equations, a considerable body of literature has been built up related to the solution of these classical field problems. Furthermore, for the nonlinear Schrödinger equation there have been several papers which propose the use of similar methods for a quantum field (Kaup 1975, Thacker 1978, Thacker and Wilkinson 1979, Creamer *et al* 1980, Sklyanin and Faddeev 1979, Sklyanin 1979, Sklyanin *et al* 1980), and these have shown some interesting connections between the operators which emerge from IST and the eigenstates of the many-particle Hamiltonian, which may be written down explicitly (McGuire 1964, Yang 1967, 1968). Thacker and Wilkinson (1979) show that IST leads to operators which create $(N+1)$-particle states from N-particle states. Similar results have been reported by Sklyanin (1979), who computed directly the effect of the operators on the N-particle Bethe eigenstates. In their latest work on the sine–Gordon equation, Sklyanin *et al* (1980) work on a lattice and choose a particular (Schrödinger) representation of the canonical quantisation rules, which avoids the problems to which we allude in this paper, although it raises new questions to which they refer.

Thacker and Wilkinson (1979) stop short of giving both the normalised creation and annihilation operators together with a *derivation* of their commutation relations, and in fact they mention some difficulties which stand in the way of determining the properties of the annihilation operators (the operators $b^*(\xi)$ of their paper). In their latest paper, Creamer *et al* (1980) appeal to the work of Sklyanin and Faddeev for the desired commutation relations rather than trying to use IST directly. We had also addressed ourselves to this problem, with the object of overcoming the difficulties found by Thacker and Wilkinson. Unfortunately our investigations using IST lead, not to the desired results, but to an impasse. This impasse is brought about by the fact that the asymptotic forms which are an essential feature of IST cannot properly be defined in the quantum field case: furthermore, if this point is ignored, then we reach a set of fundamental inconsistencies. Because of the negative conclusions which are drawn in this paper, we do not relegate technical details to an appendix since they are the central point of the paper.

2. Key results for a classical field

There is a diversity of notation and terminology in the literature on IST, so we will reiterate some of the key results in this section. We have been particularly indebted to the papers of Ablowitz *et al* (1974) and Zakharov and Shabat (1975) (referred to as AKNS and ZS respectively), but we use notation which is closer to that of Thacker and Wilkinson because of the context of our work. The nonlinear Schrödinger equation is

$$\mathrm{i}\psi_t = -\psi_{xx} + 2c^2\psi^*\psi\psi \tag{2.1}$$

and the IST method maps the unknown solution ψ onto a pair of auxiliary functions which satisfy the linear differential equations

$$v_{1x} - \tfrac{1}{2}\mathrm{i}\xi v_1 = \mathrm{i}cv_2\psi \qquad v_{2x} + \tfrac{1}{2}\mathrm{i}\xi v_2 = -\mathrm{i}c\psi^* v_1. \tag{2.2}$$

Throughout we will use lower-case letters for classical fields and the upper-case equivalents for the corresponding quantum field. Equations (2.2) were first proposed by ZS, and we will call them the ZS equations. Following AKNS (except that we use χ rather than ψ for the second independent solution of the ZS equations), we define fundamental solutions by imposing the asymptotic conditions

$$\phi \sim \begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi x/2} \\ 0 \end{pmatrix} \qquad (x \to -\infty) \tag{2.3}$$

for the first solution, and

$$\chi \sim \begin{pmatrix} 0 \\ \mathrm{e}^{-\mathrm{i}\xi x/2} \end{pmatrix} \qquad (x \to +\infty) \tag{2.4}$$

for the second. A further pair of solutions, denoted $\tilde{\phi}$ and $\tilde{\chi}$, are obtained by writing

$$\tilde{\phi} = \begin{pmatrix} \phi_2^* \\ \phi_1^* \end{pmatrix} \qquad \tilde{\chi} = \begin{pmatrix} \chi_2^* \\ \chi_1^* \end{pmatrix}. \tag{2.5}$$

All of these definitions rely on some assumption regarding the asymptotic form of ψ, which is necessary so that the right-hand side of the ZS equation may be neglected for large $|x|$.

Iteration of the ZS equations gives the fundamental solutions as

$$\begin{aligned}
\phi_1(x)\,\mathrm{e}^{-\mathrm{i}\xi x/2} &= \sum_n c^{2n} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, \mathrm{d}y_1 \ldots \mathrm{d}y_n\, \theta(x_1 < y_1 < \ldots < x_n < y_n < x) \\
&\quad \times \exp[\mathrm{i}\xi(x_1 + \ldots + x_n - y_1 \ldots - y_n)]\psi^*(x_1) \ldots \psi^*(x_n)\psi(y_1) \ldots \psi(y_n) \\
\phi_2(x)\,\mathrm{e}^{\mathrm{i}\xi x/2} &= -\mathrm{i}\sum_n c^{2n-1} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, \mathrm{d}y_1 \ldots \mathrm{d}y_{n-1}\, \theta(x_1 < y_1 \ldots < x_n < x) \\
&\quad \times \exp[\mathrm{i}\xi(x_1 + \ldots + x_n - y_1 \ldots - y_{n-1})]\psi^*(x_1) \ldots \psi^*(x_n)\psi(y_1) \ldots \psi(y_{n-1})
\end{aligned} \tag{2.6}$$

$$\chi_1(x)\,\mathrm{e}^{-\mathrm{i}\xi x/2}=-\mathrm{i}\sum_n c^{2n-1}\int \mathrm{d}x_2\ldots\mathrm{d}x_n\,\mathrm{d}y_1\ldots\mathrm{d}y_n\,\theta(x<y_1<x_2\ldots<x_n<y_n)$$
$$\times\exp[\mathrm{i}\xi(x_2+\ldots+x_n-y_1\ldots-y_n)]\psi^*(x_2)\ldots\psi^*(x_n)\psi(y_1)\ldots\psi(y_n)$$
$$\chi_2(x)\,\mathrm{e}^{\mathrm{i}\xi x/2}=\sum_n c^{2n}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\,\mathrm{d}y_1\ldots\mathrm{d}y_n\,\theta(x<x_1<y_1\ldots<x_n<y_n)$$
$$\times\exp[\mathrm{i}\xi(x_1+\ldots+x_n-y_1\ldots-y_n)]\psi^*(x_1)\ldots\psi^*(x_n)\psi(y_1)\ldots\psi(y_n).$$

The asymptotic forms of these functions define a set of 'scattering data' via

$$a(\xi)=\lim_{x\to\infty}\mathrm{e}^{-\mathrm{i}\xi x/2}\phi_1(x)\qquad b(\xi)=\lim_{x\to\infty}\mathrm{e}^{\mathrm{i}\xi x/2}\phi_2(x)\tag{2.7}$$

and explicit formulae for the coefficients $a(\xi)$ and $b(\xi)$ may be obtained by substituting the iterated solutions; we shall use this approach in the next section. It has been shown (Zakharov and Manakov 1975) that the mapping from the unknown function ψ to the scattering data is a canonical transformation when the original equation is cast in Hamiltonian form and that the new variables are action-angle variables. Thus we have a completely integrable nonlinear dynamical system, and the natural question is how this fact may be used to solve the corresponding quantum field problem.

3. Quantum fields

The quantum field equations are

$$\mathrm{i}\Psi_t=-\Psi_{xx}+2c^2\Psi^\dagger\Psi\Psi\tag{3.1}$$

together with the canonical commutation relations

$$\begin{aligned}&[\Psi(x),\Psi^\dagger(x')]=\delta(x-x')\\&[\Psi(x),\Psi(x')]=0\qquad[\Psi^\dagger(x),\Psi^\dagger(x')]=0.\end{aligned}\tag{3.2}$$

Now it is intuitively clear that we cannot use the ZS equations with the asymptotic conditions (2.3) and (2.4) to *define* auxiliary quantum fields, since the commutation relations (3.2) preclude the assumption that Ψ is negligible for sufficiently large $|x|$. Of course, the effect of the operator Ψ on any particular state vector may become negligible in this limit, but this is not equivalent to the assumption that the operator approaches the zero operator. This is precisely the difference between strong and weak convergence for operators (Bachman and Narici 1966), and Sklyanin notes that his asymptotic conditions are to be understood in the weak sense. We will return to this point; for the moment we avoid it by using the iterative solutions (2.6) to define the auxiliary fields which were written there so that the substitution $\psi\to\Psi$ leads to normal ordering. $A(\xi)$ and $B(\xi)$ are defined similarly:

$$A(\xi)=\sum_n c^{2n}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\,\mathrm{d}y_1\ldots\mathrm{d}y_n\,\theta(x_1<y_1\ldots<x_n<y_n)$$
$$\times\exp[\mathrm{i}\xi(x_1+\ldots+x_n-y_1\ldots-y_n)]\Psi^\dagger(x_1)\ldots\Psi^\dagger(x_n)\Psi(y_1)\ldots\Psi(y_n)\tag{3.3}$$

$$B(\xi)=-\mathrm{i}\sum_n c^{2n-1}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\,\mathrm{d}y_1\ldots\mathrm{d}y_{n-1}\,\theta(x_1<y_1\ldots<y_{n-1}<x_n)$$
$$\times\exp[\mathrm{i}\xi(x_1+\ldots+x_n-y_1\ldots-y_{n-1})]\Psi^{\dagger}(x_1)\ldots\Psi^{\dagger}(x_n)\Psi(y_1)\ldots\Psi(y_{n-1}). \tag{3.4}$$

Consider now the commutators of $A(\xi)$ and $B(\xi)$ with an arbitrary operator Λ. Looking first at $A(\xi)$, the nth term in the expansion depends on

$$[\Psi^{\dagger}(x_1)\ldots\Psi^{\dagger}(x_n)\Psi(y_1)\ldots\Psi(y_n),\Lambda]. \tag{3.5}$$

As was observed by Thacker and Wilkinson, the ordering introduced by the multiple step function enables us to write the commutator as

$$[\Psi^{\dagger}(x_1)\Psi^{\dagger}(x_2)\Psi(y_1)\Psi^{\dagger}(x_3)\Psi(y_2)\ldots\Psi(y_{n-1})\Psi(y_n),\Lambda] \tag{3.6}$$

and in evaluating $[\Psi^{\dagger}(x_j)\Psi(y_{j-1}),\Lambda]$ we may use the fact that $[\Psi(x),\Psi^{\dagger}(x')]$ is a c-number, together with the Jacobi identity, to write

$$[\Psi^{\dagger}(x_j)\Psi(y_{j-1}),\Lambda]=\Psi(y_{j-1})[\Psi^{\dagger}(x_j),\Lambda]+[\Psi(y_{j-1}),\Lambda]\Psi^{\dagger}(x_j). \tag{3.7}$$

Hence if we define $F(x)$ and $G(x)$ by

$$[\Psi(x),\Lambda]=F(x)\qquad [\Psi^{\dagger}(x),\Lambda]=G(x) \tag{3.8}$$

we find that the commutator (3.5) is

$$\begin{aligned}
&\Psi^{\dagger}(x_1)F(y_1)\Psi^{\dagger}(x_2)\ldots\Psi^{\dagger}(x_n)\Psi(y_2)\ldots\Psi(y_n)\\
&\quad+\Psi^{\dagger}(x_1)\Psi^{\dagger}(x_2)\Psi(y_1)F(y_2)\Psi^{\dagger}(x_3)\ldots\Psi^{\dagger}(x_n)\Psi(y_3)\ldots\Psi(y_n)\\
&\quad+\ldots\\
&\quad+\Psi^{\dagger}(x_1)\ldots\Psi^{\dagger}(x_n)\Psi(y_1)\ldots\Psi(y_{n-1})F(y_n)\\
&\quad+G(x_1)\Psi^{\dagger}(x_2)\ldots\Psi^{\dagger}(x_n)\Psi(y_1)\ldots\Psi(y_n)\\
&\quad+\Psi^{\dagger}(x_1)\Psi(y_1)G(x_2)\Psi^{\dagger}(x_3)\ldots\Psi^{\dagger}(x_n)\Psi(y_2)\ldots\Psi(y_n)\\
&\quad+\ldots\\
&\quad+\Psi^{\dagger}(x_1)\ldots\Psi^{\dagger}(x_{n-1})\Psi(y_1)\ldots\Psi(y_{n-1})G(x_n)\Psi(y_n).
\end{aligned} \tag{3.9}$$

A tedious but straightforward calculation shows that when we substitute this into the integral (3.3) we have all of the terms of order c^{2n} in the expansion of an integral involving the auxiliary functions, consequently

$$[A(\xi),\Lambda]=\mathrm{i}c\int \mathrm{d}x\,(\Phi_2(x)F(x)X_2(x)+\Phi_1(x)G(x)X_1(x)). \tag{3.10}$$

All of this was noted by Thacker and Wilkinson, except that they restricted the operator Λ to one of $A(\xi)$, $B(\xi)$, $A^{\dagger}(\xi)$ or $B^{\dagger}(\xi)$ so as to write their results in terms of functional derivatives, but this is not necessary and leads to undue complication.

For the operator $B(\xi)$, (3.9) is replaced by

$$\begin{aligned}
&\Psi^\dagger(x_1)F(y_1)\Psi^\dagger(x_2)\ldots\Psi^\dagger(x_n)\Psi(y_2)\ldots\Psi(y_{n-1})\\
&\quad+\Psi^\dagger(x_1)\Psi^\dagger(x_2)\Psi(y_1)F(y_2)\Psi^\dagger(x_3)\ldots\Psi^\dagger(x_n)\Psi(y_3)\ldots\Psi(y_{n-1})\\
&\quad+\ldots\\
&\quad+\Psi^\dagger(x_1)\ldots\Psi^\dagger(x_{n-1})\Psi(y_1)\ldots\Psi(y_{n-2})F(y_{n-1})\Psi^\dagger(x_n)\\
&\quad+G(x_1)\Psi^\dagger(x_2)\ldots\Psi^\dagger(x_n)\Psi(y_1)\ldots\Psi(y_{n-1})\\
&\quad+\Psi^\dagger(x_1)\Psi(y_1)G(x_2)\Psi^\dagger(x_3)\ldots\Psi^\dagger(x_n)\Psi(y_2)\ldots\Psi(y_{n-1})\\
&\quad+\ldots\\
&\quad+\Psi^\dagger(x_1)\ldots\Psi^\dagger(x_{n-1})\Psi(y_1)\ldots\Psi(y_{n-1})G(x_n)
\end{aligned}\tag{3.11}$$

and again a straightforward calculation leads to the formula

$$[B(\xi),\Lambda]=-\mathrm{i}c\int \mathrm{d}x\;(\Phi_2(x)F(x)X_1^\dagger(x)+\Phi_1(x)G(x)X_2^\dagger(x)).\tag{3.12}$$

This result calls for some comment, since our conclusions differ from those of Thacker and Wilkinson. As they observe, there are terms in (3.11), when $G(x)$ is non-zero, involving the operator products

$$\Psi^\dagger(x_{m+1})\ldots\Psi^\dagger(x_n)\Psi(x_m)\ldots\Psi(x_{n-1})\tag{3.13}$$

and for these the operator ordering does not coincide with the ordering of the variables in the step function in (3.4). However, our result is correct, because the adjoints of X_1 and X_2 are

$$\begin{aligned}
X_1^\dagger(x)\,\mathrm{e}^{\mathrm{i}\xi x/2}&=\mathrm{i}\sum_n c^{2n-1}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\,\mathrm{d}y_2\ldots\mathrm{d}y_n\\
&\quad\times\theta(x<x_1<y_2<x_2<\ldots<y_n<x_n)\exp[\mathrm{i}\xi(x_1+\ldots+x_n-y_2-\ldots-y_n)]\\
&\quad\times\Psi^\dagger(x_1)\ldots\Psi^\dagger(x_n)\Psi(y_2)\ldots\Psi(y_n)\\
X_2^\dagger(x)\,\mathrm{e}^{-\mathrm{i}\xi x/2}&=\sum_n c^{2n}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\,\mathrm{d}y_1\ldots\mathrm{d}y_n\\
&\quad\times\theta(x<y_1<x_1\ldots<y_n<x_n)\exp[\mathrm{i}\xi(x_1+\ldots+x_n-y_1-\ldots-y_n)]\\
&\quad\times\Psi^\dagger(x_1)\ldots\Psi^\dagger(x_n)\Psi(y_1)\ldots\Psi(y_n)
\end{aligned}\tag{3.14}$$

after we have interchanged the meanings of the variables x_i and y_j so as to retain the x's for the creation operators and the y's for the annihilation operators for use with (3.11). This change of variable names, which is necessitated by the fact that the creation and annihilation operators are interchanged when we take the adjoint, brings about precisely the ordering which is needed for the commutation relation (3.12) to work out correctly.

On setting Λ to Ψ and $\Psi^\dagger$, we find immediately that

$$\begin{aligned}
&[A(\xi),\Psi(x)]=-\mathrm{i}c\Phi_1(x)X_1(x) &&[B(\xi),\Psi(x)]=\mathrm{i}c\Phi_1(x)\tilde{X}_1(x)\\
&[A(\xi),\Psi^\dagger(x)]=\mathrm{i}c\Phi_2(x)X_2(x) &&[B(\xi),\Psi^\dagger(x)]=-\mathrm{i}c\Phi_2(x)\tilde{X}_2(x).
\end{aligned}\tag{3.15}$$

4. The auxiliary fields

The fundamental results (3.10) and (3.12) involve the use of the auxiliary fields. We have introduced these fields by the definitions (2.6), from which we may show that they satisfy the normal ordered zs equation

$$(\mathrm{d}/\mathrm{d}x - \tfrac{1}{2}\mathrm{i}\xi)\Phi_1 = \mathrm{i}c\Phi_2\Psi \qquad (\mathrm{d}/\mathrm{d}x + \tfrac{1}{2}\mathrm{i}\xi)\Phi_2 = -\mathrm{i}c\Psi^\dagger\Phi_1 \tag{4.1}$$

with the same for $\tilde{\Phi}$, X, and $\tilde{X}$. It follows also from (2.6) that

$$\left.\begin{aligned}[\Phi_i(x), \Psi(x')] = 0\\ [\Phi_i(x), \Psi^\dagger(x')] = 0\end{aligned}\right\} \qquad (x' > x) \tag{4.2}$$

and

$$\left.\begin{aligned}[X_i(x), \Psi(x')] = 0\\ [X_i(x), \Psi^\dagger(x')] = 0\end{aligned}\right\} \qquad (x' < x). \tag{4.3}$$

Now we wish to set $x = x'$ in these commutators and, as Thacker and Wilkinson have shown, this is easy to do once we have assigned a meaning to the integral

$$\int \mathrm{d}x\, \theta(x)\delta(x). \tag{4.4}$$

The symmetric choice is one-half, leading to the following non-zero commutators:

$$\begin{aligned}&[\Phi_2(x), \Psi(x)] = \tfrac{1}{2}\mathrm{i}c\Phi_1(x) \qquad &&[\Phi_1(x), \Psi^\dagger(x)] = \tfrac{1}{2}\mathrm{i}c\Phi_2(x)\\ &[X_2(x), \Psi(x)] = -\tfrac{1}{2}\mathrm{i}cX_1(x) \qquad &&[X_1(x), \Psi^\dagger(x)] = -\tfrac{1}{2}\mathrm{i}cX_2(x).\end{aligned} \tag{4.5}$$

Observe that this result implies that we cannot replace the auxiliary fields by 'asymptotic forms' for large $|x|$. Indeed, on using these commutators, we may rewrite the zs equations in anti-normal ordered form as

$$(\mathrm{d}/\mathrm{d}x - \tfrac{1}{2}\mathrm{i}\xi + \tfrac{1}{2}c^2)\Phi_1 = \mathrm{i}c\Psi\Phi_2 \qquad (\mathrm{d}/\mathrm{d}x - \tfrac{1}{2}\mathrm{i}\xi - \tfrac{1}{2}c^2)\Phi_2 = -\mathrm{i}c\Phi_1\Psi^\dagger \tag{4.6}$$

and now it would appear that the asymptotic forms have a different x-dependence, which conflicts with a relation like

$$\left.\begin{aligned}\Phi_1(x) \sim A(\xi)\,\mathrm{e}^{\mathrm{i}\xi x/2}\\ \Phi_2(x) \sim B(\xi)\,\mathrm{e}^{-\mathrm{i}\xi x/2}\end{aligned}\right\} \qquad (x \to +\infty) \tag{4.7}$$

since the operators $A(\xi)$ and $B(\xi)$ are supposed to be independent of x, and only they are affected by replacing normal ordering by anti-normal ordering. In a similar way, we may analyse Sklyanin's proposal to extract the commutation relations of $A(\xi)$ and $B(\xi)$ from a similarity argument between the differential equations for the vectors

$$H_1(x) = \begin{pmatrix}\Phi_1(x, \xi)\Phi_1(x, \xi')\\ \Phi_1(x, \xi)\Phi_2(x, \xi')\\ \Phi_2(x, \xi)\Phi_1(x, \xi')\\ \Phi_2(x, \xi)\Phi_2(x, \xi')\end{pmatrix} \tag{4.8}$$

and

$$H_2(x) = \begin{pmatrix}\Phi_1(x, \xi')\Phi_1(x, \xi)\\ \Phi_2(x, \xi')\Phi_1(x, \xi)\\ \Phi_1(x, \xi')\Phi_2(x, \xi)\\ \Phi_2(x, \xi')\Phi_2(x, \xi)\end{pmatrix}. \tag{4.9}$$

We will not go into details here, but simply note that if we replace H_1 and H_2 by the supposed asymptotic forms which follow from (4.7), then the resulting expressions do not satisfy the differential equations which are obtained by neglecting terms in $\Psi(x)$ and $\Psi^\dagger(x)$. One possible reaction to this situation is to go back to (4.4), and make a different choice or, equivalently, alter the relations (4.5). However, it is not difficult to show that the only choice which avoids the difficulties is to set all of the commutators (4.5) to zero. Then Sklyanin's arguments do indeed work, to show that everything commutes which is useless for a quantum field theory.

Turning to Thacker and Wilkinson's approach utilising equations (3.10) and (3.12), we may set $\Lambda = A(\xi')$ in (3.10) to find that

$$[A(\xi), A(\xi')] = c^2 \int dx\, (\Phi_1\Phi_2'X_2'X_1 - \Phi_2\Phi_1'X_1'X_2) \tag{4.10}$$

where $\Phi_i = \Phi_i(x, \xi)$, $\Phi_i' = \Phi_i(x, \xi')$, etc. Now we can use the commutation relations (4.5) together with the ZS equation to show that

$$\frac{d}{dx}(\Phi_1X_2' - \Phi_2X_1')(\Phi_1'X_2 - \Phi_2'X_1) = -i(\xi - \xi')(\Phi_1X_2'\Phi_2'X_1 - \Phi_2X_1'\Phi_1'X_2) \tag{4.11}$$

which is also valid if Φ_i and/or X_i is replaced by $\tilde{\Phi}_i$ and $\tilde{X}_i$ respectively. This means that we may *formally* write (4.10) as

$$[A(\xi), A(\xi')] = \frac{ic^2}{\xi - \xi'}[(\Phi_1X_2' - \Phi_2X_1')(\Phi_1'X_2 - \Phi_2'X_1)]_{-\infty}^{\infty}. \tag{4.12}$$

Applying the same argument to (3.12) gives

$$[B(\xi), A(\xi')] = -\frac{ic^2}{\xi - \xi'}[(\Phi_1X_2' - \Phi_2X_1')(\Phi_1'\tilde{X}_2 - \Phi_2'\tilde{X}_1)]_{-\infty}^{\infty} \tag{4.13}$$

and again, on setting $\Lambda = B(\xi')$, we find that

$$[A(\xi), B(\xi')] = -\frac{ic^2}{\xi - \xi'}[(\Phi_1\tilde{X}_2' - \Phi_2\tilde{X}_1')(\Phi_1'X_2 - \Phi_2'X_1)]_{-\infty}^{\infty} \tag{4.14}$$

$$[B(\xi), B(\xi')] = \frac{ic^2}{\xi - \xi'}[(\Phi_1\tilde{X}_2' - \Phi_2\tilde{X}_1')(\Phi_1'\tilde{X}_2 - \Phi_2'\tilde{X}_1)]_{-\infty}^{\infty}. \tag{4.15}$$

Assuming that we may use asymptotic forms in (4.12) leads simply to the conclusion that

$$[A(\xi), A(\xi')] = 0. \tag{4.16}$$

When we attempt the same argument for (4.13) and (4.14), we have the apparent slight complication that one of the terms oscillates as $|x| \to \infty$, but this only indicates the presence of a delta function, exactly as when the classical Poisson brackets are evaluated (Zakharov and Manakov 1975). Setting this aside by restricting our attention to the case that $\xi \neq \xi'$, we have

$$[B(\xi), A(\xi')] = -\frac{ic^2}{\xi - \xi'}A(\xi')B(\xi) \tag{4.17}$$

and

$$[A(\xi), B(\xi')] = -\frac{ic^2}{\xi - \xi'}B(\xi')A(\xi) \tag{4.18}$$

and at this point we arrive at a contradiction, since the two results together imply that

$$\frac{\xi-\xi'+\mathrm{i}c^2}{\xi-\xi'}=\frac{\xi-\xi'}{\xi-\xi'-\mathrm{i}c^2} \tag{4.19}$$

which can only be true in the uninteresting case of no interaction ($c=0$).

5. Discussion and conclusions

In the classical field theory, IST may be used *ab initio* to provide a complete solution of the equations under very general boundary conditions. For example, in the present case the Poisson brackets can be evaluated in a direct manner (Zakharov and Manakov 1975), and in fact Thacker and Wilkinson's program is essentially an attempt to transcribe these methods into the quantum field domain. The classical canonical transformation provides a new set of canonical variables in terms of which the modes of motion are independent but, as Kaup (1975) has explicitly verified, the use of canonical quantisation rules

$$[q_i, p_i]=\mathrm{i} \tag{5.1}$$

on these new variables is not equivalent to the quantisation rules which are embodied in (3.2). It is not clear to us why IST leads to an impasse in the quantum case, but it may be related to the well known fact there are are an infinite number of non-unitarily equivalent representations of the canonical quantisation rules for systems with an infinite number of degrees of freedom (Prugovecki 1971). This is a separate problem to the above-mentioned fact that the nonlinear transformations provided by IST do not preserve canonical quantisation.

Returning to Sklyanin's paper, he extracts the commutation relations by defining $A(\xi)$ and $B(\xi)$ via equations equivalent to (3.3) and (3.4), and then considering their effect on the N-particle Bethe eigenstates

$$\phi_N(x_1,\ldots,x_N)=\frac{1}{(N!)^{1/2}}\sum_{(l_1\ldots l_N)}\prod_{r<s}\left(1+\frac{c^2}{\mathrm{i}(k_{l_r}-k_{l_s})}\right)\exp\left(\mathrm{i}\sum_j k_{l_j}x_j\right)$$
$$(x_1>x_2\ldots>x_N). \tag{5.2}$$

Since these functions are known to diagonalise the Hamiltonian, this procedure must be sound, although it suffers from the disadvantage that the solution to the corresponding N-body problem is needed before the correct commutation relations may be derived. The alternative approach (Sklyanin *et al* 1980) is to replace the ZS equations by a finite difference approximation over a finite interval, so that the canonical commutation relations may be readily realised, first at each lattice site, and then globally by employing finite cartesian products. An important feature of this approach is that it does not suffer from the problem of treating the fields as operators rather than operator-valued distributions, since there are only a finite number of sites at every stage of the investigation. Regrettably, when the infinite limits are taken on these equations, the inconsistency between the commutators (4.17) and (4.18) reappears. An advantage of this method is that it clearly demonstrates the impossibility of asymptotic relations like (4.7).

What we would like, however, is to be able to use at least some part of the classical theory of IST and associated methods *directly* in the quantum domain, without fear of inconsistency. Creamer, Thacker and Wilkinson have demonstrated that such a theory is of immense utility. One possibility, which is at present under investigation, is to use the iterative methods of Rosales (1978) in a direct way. Essentially, Rosales showed that the iterative solutions of the equations commonly investigated with IST are all comparatively simple when the Fourier transform is employed. Specifically, he showed that the solution of the nonlinear Schrödinger equation with a repulsive potential is of the form

$$\psi(x) = \int \mathrm{d}\xi_1\, \rho(\xi_1) \exp(\mathrm{i}\xi_1 x - \mathrm{i}\xi_1^2 t) + c^2 \int \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3 \times \frac{\rho(\xi_1)\rho^*(\xi_2)\rho(\xi_3) \exp[\mathrm{i}(\xi_1 - \xi_2 + \xi_3)x - \mathrm{i}(\xi_1^2 - \xi_2^2 + \xi_3^3)t]}{(\xi_2 - \xi_1 - \mathrm{i}\epsilon)(\xi_3 - \xi_2 + \mathrm{i}\epsilon)} + \ldots \tag{5.3}$$

where the function $\rho(\xi)$ is arbitrary apart from general restrictions. In its normal ordered form, this equation is one of the two key results of Creamer, Thacker and Wilkinson, once we replace $\rho(\xi)$ by $R(\xi)$ and adopt a normal ordering. We believe that this direct approach will lead to a theory with all the power and elegance of the work of Creamer, Thacker and Wilkinson, but without the disadvantage of needing to discuss the asymptotic behaviour of the auxiliary functions. The results of this investigation will be published separately when it is concluded.

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